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A SIMPLE DIRECTION MODEL OF ELECTORAL COMPETITION*

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* I wish to thank John Ferejohn, Morris Fiorina, Melvin Hinich, and Charles Plott for their valuable criticism, comments and time.

Social Science Working Paper

Number 128

June 1976

Revised September 1977

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Since the seminal contribution of Downs [1957], spatial models have been used to analyze the electoral process. However, their utility has been severely limited by (at least) four stringent assumptions.[1] First, typical spatial models, henceforth to be called Euclidean models, require that the messages candidates transmit to voters be the points of an Euclidean issue space. A point message indicates a candidate's promised issue outcome. Perfect candidate mobility and a perfect flow of information from candidates to voters are two aspects of this assumption. Secondly, in the basic spatial models all promises are believed -- the issue outcome that a voter believes will occur if a candidate is elected is assumed to be identical to the candidate's point message. Thirdly, every individual's preferences are required to be complete over the entire issue space and often to decline with distance from an ideal point. Finally, candidates are usually assumed to perceive the preferences of all voters over all points in the issue space.

These requirements of Euclidean spatial models have been questioned by political scientists -- Page [1975] is particularly critical. In this paper, a weakening of each of the above assumptions will be shown to lead naturally to a model employing a non-Euclidean outcome space which can be viewed as the set of points on the surface of a hypersphere. Under the primary interpretations to be offered in section I, this space is composed of the directions in which a status quo point in an Euclidean issue space can shift.

In section II the basic model is described as a two-person plurality game in which the candidates adopt shift directions as strategies. Equilibrium directions in this game, however, are shown to be in the core of a corresponding n-person absolute majority rule game. Necessary and sufficient conditions are then easily established for the existence of an equilibrium direction.

In section III, optimal strategies for a candidate competing against a rigid opponent are investigated. The result is a prediction of candidate divergence, somewhat analogous to that made by Hinich and Ordeshook [1968] within the context of an Euclidean model.

Finally, directional voting is embedded into the framework of Euclidean models in section IV, and the existence of point equilibria is shown to imply the existence of equilibrium directions. Equilibrium direction vectors will be shown to "point" towards equilibrium points, provided the latter exist.

I. MOTIVATIONS AND ASSUMPTIONS

Four different conceptualizations of the set of messages that candidates send to voters, the set of possible outcomes that

voters perceive, and the relationship between these two sets can serve as foundations to the basic direction model. First, both the messages candidates transmit and the outcomes voters associate with them can be considered as single points in an Euclidean issue space. However, individuals may often map all candidate messages into point outcomes only a marginal distance from the status quo -- the point in the issue space that represents the current state of the world on the relevant issues. The possible causes of this virtual shrinkage of the issue space are twofold: (1) for physical or political reasons, candidate mobility in the message space may be restricted to a neighborhood of the status quo -- truthful and knowledgeable candidates will only choose messages within this neighborhood; (2) based perhaps on past performances, voters may not believe any winning candidate can achieve a large shift of the status quo, regardless of campaign promises (messages). When a candidate's actions can only marginally shift the status quo, only the directions in which he proposes to shift it are important. Strategies can be considered as directions which shall be represented as vectors of unit or zero length.

A second behavioral motivation of the direction model can be based on imperfect communication. Candidates may still attempt to send messages that voters will view as point outcomes. But due to high information costs, voters may not become aware of the exact issue positions that candidates adopt. From Campbell et al. [1960] to Page [1975], empirically-oriented political scientists have been critical of models that assume a perfect flow of information from candidates to voters. However, if candidates are able to at least convey their pro and con opinions and the relative stresses they place upon the issues, they may be able to transmit the directions in which they would shift the status quo.

Thirdly, suppose one of the following is true: (1) as Page [1975] suggests, individual preference ordering are complete or well-defined only in a neighborhood of the familiar status quo; (2) individual indifference surfaces actually take the form of rays emanating from the status quo; or (3) candidates only receive reliable information about preferences near the status quo. Then candidates may have no incentive to adopt more than directions or, equivalently, marginally shifted points as their strategies, since they can know only how voters respond to such strategies.

The above rationalizations for direction strategies have been based upon the concept of an Euclidean issue space. However, if the outcome space into which voters transform candidate messages is cognitive or perceptual in nature, it may not possess the Euclidean structure. In particular, Weisberg [1974] hypothesizes that some political issue spaces can be modeled as closed circles. As an example, Weisberg refers to the Swedish Riksdag, where parties of the so-called left and right sometimes vote together against the moderates. So the fourth conceptualization that can serve as a basis for the direction model, although it would now be inappropriately named, is that the set of perceived outcomes is a non-Euclidean space isomorphic to the surface of a hypersphere.[2]

Assumptions about individual preferences are also required. In the basic model we assume each voter most prefers the status quo to shift in a particular direction. A voter will rank directions negatively with the size of the angle they form with his most preferred direction. Formally, suppose that v_1 and v_2 are two direction vectors, and s is the direction vector representing some voter's most preferred direction. Then the voter will prefer the direction of v_1 to that of v_2 if and only if $s'v_1 > s'v_2$. Furthermore, as is usual, we assume a voter's preferences for candidates are identical to his preferences for the directions they adopt.

All of these preference assumptions are analogous to those made in simple Euclidean spatial models -- simply substitute preferred points for preferred directions, and Euclidean distances for angles. But they can be better justified here.[3] Suppose two candidates choose vectors z_1 and z_2 that are the same distance d from the status quo in the directions of $v_1 = z_1/d$ and $v_2 = z_2/d$. Then the directional preferences described above approximate preferences that can be represented by a differentiable utility function -- $s'(v_1 - v_2)$ is a linear approximation to $[u(z_1) - u(z_2)]/d$ when s is the (normalized) utility gradient at the status quo. The approximation becomes exact if candidates can adopt points only marginally distinct from the status quo.[4]

II. THE BASIC MODEL

In the basic direction model, two candidates compete by choosing vectors v_1 and v_2 of unit or, to allow null shifts, zero length in the set of <u>directions</u> $\mathcal{B} = B \cup \{0\}$, where $B = \{v \in E^n : \| v \| = 1\}$. Each voter i most prefers a vector $s_i \in \mathcal{B}$. An arbitrary probability measure P defined on (Borel) subsets of $\mathcal{B}$ represents the distribution of voters' preferred direction vectors, imposing no limitation on the number of voters. The directional preferences of voter i are represented by the inner product $s_i'v$. Thus the fraction of the electorate who votes for candidate j is $P[s'(v_j - v_k) > 0]$. Geometrically, for the case of $v_j \neq 0$ ($j = 1, 2$), j's votes are obtained from the fraction of the electorate whose preferred direction vectors lie on the same side as v_j of a hyperplane containing the origin and the mid-vector $v_1 + v_2$. The indifferent voters are those whose ideal direction vectors lie in this dividing hyperplane -- for lack of a more realistic assumption in this setting, they are assumed to abstain. Notice that voters with $s_i = 0$ are assumed to always be indifferent.

Each candidate j is assumed to maximize his plurality:

$$PL_j(v_1,v_2) = P[s'(v_j - v_k) > 0] - P[s'(v_j - v_k) < 0].$$

Because of the symmetry of the two person game played by the candidates, an equilibrium can be defined as a direction that guarantees a nonnegative plurality to any candidate who adopts it.

<u>Definition 1</u>: An <u>equilibrium direction vector</u> v^* is a direction in $\mathcal{B}$ for which $PL_1(v^*, v) \geq 0$ for all $v \in \mathcal{B}$.

The first task is to show the relationship between equilibrium directions in the two-person plurality game and undominated directions in the n-person absolute majority game. An undominated direction in the latter is one that is not ranked below another by a strict majority of the voters[5]:

<u>Definition 2</u>: A direction vector $v^* \in \mathcal{B}$ is <u>undominated</u> provided $P[s'(v^* - v)] \geq 1/2$ for all $v \in \mathcal{B}$.

It would be disturbing to find equilibrium directions that were not undominated, for then a direction may exist which is preferred by a majority to the direction adopted by the winning candidate. Theorem 1 below shows that this cannot occur. Furthermore, theorem 1 shows that undominated directions are equilibria if $P[s = 0] = 0$, that is, if nobody is indifferent over all directions. This result is not obvious because a positive fraction of the voters may still be indifferent between any two directions v_1 and v_2, allowing the possibility that $P[s'(v_1 - v_2)] \geq 1/2$ even though $PL_1(v_1, v_2) < 0$. (The lengthy proof of theorem 1 is in an Appendix.)

<u>Theorem 1:</u> Equilibrium directions are undominated. Conversely, if $P[s = 0] = 0$, then undominated directions are equilibrium directions.

One use of theorem 1 is to provide necessary conditions for equilibrium directions, since a condition both necessary and sufficient for undominated directions is easily obtained.

Theorem 2, similar to a result provided in Matthews (1977) for a finite set of voters, is proved below in general and then discussed.

Theorem 2: v^* is an undominated direction vector if and only if $P[s'a \geq 0] \geq 1/2$ for all $a \in E^n$ satisfying $a'v^* \geq 0$.

Proof: Suppose v^* is undominated and that $a'v^* > 0$. We may assume $a = 1$, and hence, letting $v = v^* - (2a'v^*)a$, have that $v \in B$. Then $a'v^* > 0$ and $P[s'a \geq 0] = P[(2a'v^*)s'a \geq 0] = P[s'(v^* - v) \geq 0] \geq 1/2$. If $a'v^* = 0$ and $v^* \neq 0$, there exists a sequence $\{a_1, a_2, \ldots\}$ that converges to a and whose members satisfy $a_n'v^* > 0$. Hence $P[s'a_n \geq 0] \geq 1/2$ for all a_n, and $P[s'a \geq 0] \geq \lim_{n\to\infty} P[s'a_n \geq 0] \geq 1/2$ is established by an argument like that used to prove theorem 1. Finally, if $v^* = 0$, then $P[s'a \geq 0] = P[s'(v^* -(-a) \geq 0] \geq 0] \geq 1/2$ for any $a \in B$ and hence for any $a \in E^n$.

Conversely, suppose $P[s'a \geq 0] \geq 1/2$ whenever $a'v^* \geq 0$. Since $(v^* - v)'v^* \geq 0$ for any $v \in B$, $P[s'(v^* - v)] \geq 1/2$ and v^* is dominant.

The condition of theorem 2 actually consists of two different parts, namely, that $P[s'a \geq 0] \geq 1/2$ whenever (1) $a'v^* = 0$ and

whenever (2) $a'v^* > 0$. Satisfaction of the first part means simply that the individuals whose ideal direction vectors lie upon any hyperplane containing v^* and the origin, or to one side of it, constitute a (weak) majority of all individuals. This property is entirely analogous to the property that Hoyer and Mayer [1974, 1975] define a total median to satisfy for an Euclidean spatial model: every hyperplane containing a total median must bisect the distribution of voters' preferred points in the issue space.[6]

Davis, DeGroot, and Hinich [1972], and later Sloss [1973] and Hoyer and Mayer [1974, 1975], show that in the simple Euclidean model an undominated point exists if and only if it is a total median. But for the direction model, part (2) as well as part (1) of the condition in theorem 2 is needed to obtain existence. Distributions of the electorate exist that satisfy the bisecting property of part (1), but do not allow the existence of undominated directions. A continuous example appears in figure 1, where the distribution of preferred directions is represented by the area between the unit circle B and the curve f(s). Each of the lines M_1, M_2, and M_3 has a greater fraction of the electorate's preferred directions on one side of it than on the other. (The signs "+" and "-" near each line M_i indicate which side of it the greater fraction of voters' preferred directions lie.) No undominated direction can exist, since any direction will lie on the "-" side of some line M_i and so will receive fewer votes than a direction located symmetrically on the opposite side of M_i. However, some

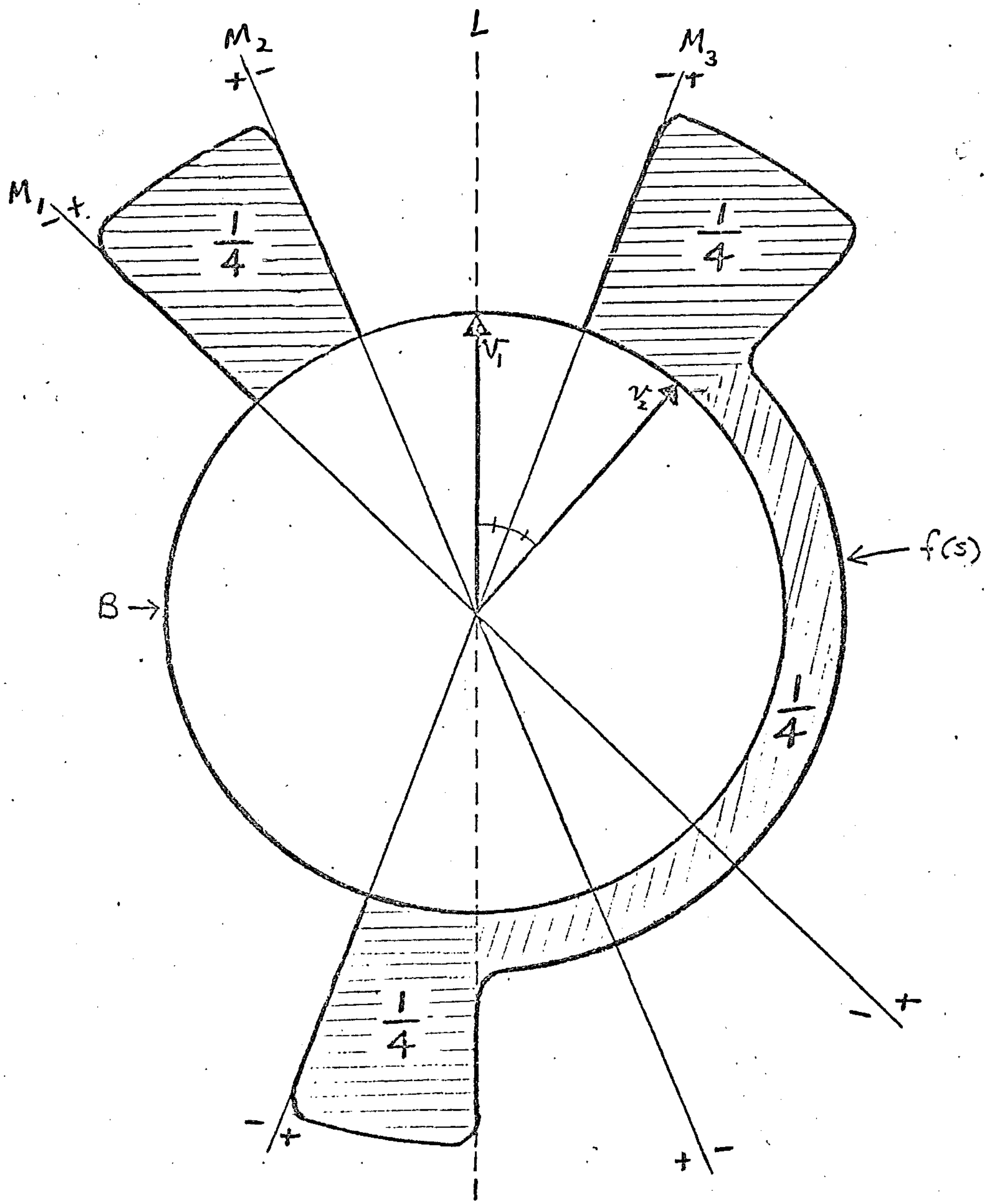

FIGURE 1

A Distribution in Which No Direction Is an Equilibrium Even Though a Bisecting Direction Vector Exists

directions will satisfy part (1) of the condition, such as the vector v_1 that lies in the bisecting line L. Since v_1 lies on the "-" side of M_1, it will receive only 1/4 the votes in a contest against v_2.

In a Euclidean model, a candidate who diverges from a fixed opponent will only lose votes. So if the opponent has chosen a total median, the diverging candidate can only decrease his plurality from zero. In the direction model, however, a diverging candidate will gain the votes of voters whose preferred directions directly oppose those of the voters he loses. Even if the opponent has adopted a median-like direction, a diverging candidate may win a strict majority by diverging so as to gain more votes than he loses. The complete condition of theorem 2 eliminates this possibility for an undominated v^* by requiring a majority to have its preferred directions on the same side as v^* of any hyperplane containing the origin.

It can now be shown that the zero direction is an equilibrium or is undominated if and only if the same is true of all directions. Interpreted loosely, this means that a proposal to not shift the status quo is winning if and only if any other proposed shift is also winning. One could say in this case that society is indifferent as to the direction the status quo marginally shifts, just as an individual would be if the status quo were located at an extremum of his utility function.

<u>Corollary 1</u>: The zero direction is an equilibrium (undominated) if and only if all directions are equilibria (undominated).

Proof: By theorem 2, $0 \varepsilon B$ is undominated iff $P[s'a \geq 0] \geq 1/2$ for all $a \varepsilon E^n$, which is true iff all $v \varepsilon B$ are undominated. 0 is an equilibrium provided $PL_1(0,v) \geq 0$ for all $v \varepsilon B$, which is true iff $P[s'v < 0] = P[s'v > 0]$ for all $v \varepsilon B$. But the latter is true iff $PL_1(v_1, v_2) = 0$ for all $v_1, v_2 \varepsilon B$, or rather, iff every $v \varepsilon B$ is an equilibrium.

III. EXPLOITING A FIXED OPPONENT

In this section we show that if one candidate rigidly adopts a direction vector on a particular side of B, to be called B^-, then the optimal vector for the opponent to choose lies in B^+, the side of B opposite B^-. The two vectors will be located symmetrically about the hyperplane that separates B^+ from B^-. Thus, entirely half the directions will be inferior in the sense that only if both candidates are rigid will they both choose inferior directions. Since B^+ shall be defined as the half of B containing the largest fraction of non-indifferent voters, this result may also be interpreted as follows: once an extremist candidate becomes too extreme, the more extreme he becomes the further his opponent should diverge from him. Although this divergence result is similar to that which Hinich and Ordeshook [1968] proved for Euclidean models, it differs fundamentally by not requiring abstention of nonindifferent voters. Furthermore, no symmetry requirements are imposed or equilibriums assumed to exist.

Before formally presenting theorem 3, we need some definitions.

Definition 3: Let $\overline{P} = \sup\{P[s'c > 0] - P[s'c < 0]\}$. Assuming a vector $\overline{c} \varepsilon B$ exists such that $\overline{P} = P[s'\overline{c} > 0] - P[s'\overline{c} < 0]$, let $B^- = \{v \varepsilon B: v'\overline{c} < 0\}$, and $B^+ = \{v \varepsilon B: v'\overline{c} \geq 0\}$.

The direction vector $\overline{c}$ exists if P represents either a continuum of voters or a finite number of voters. Hence it is not restrictive to assume for the remainder of this section that $\overline{c}$ exists. The interesting case is when $\overline{P} > 0$, in which case the following also indicates that any equilibrium direction vector is in B^+.

Theorem 3: If $v_2 \varepsilon B^-$, then the function $f(v) = PL_1(v, v_2)$ is maximized on B by a vector $\overline{v} = v_2 - (2\overline{c}' v_2)\overline{c}$ contained in B^+.

Proof: Clearly $\| \overline{v} \| = 1$ and $\overline{v}'\overline{c} = -v_2'\overline{c} > 0$. Hence $\overline{v} \varepsilon B^+$. The proof is finished by observing that

$$
\begin{aligned}
PL_1(\overline{v}, v_2) &= P[s'(\overline{v} - v_2) > 0] - P[s'(\overline{v} - v_2) < 0] \\
&= P[(-2\overline{c}'v_2)s'\overline{c} > 0] - P[(-2\overline{c}'v_2)s'\overline{c} < 0] \\
&= P[s'\overline{c} > 0] - P[s'\overline{c} < 0] \\
&= \overline{P}.
\end{aligned}
$$

Theorem 3 is illustrated in figure 2, which also indicates further results obtainable when P exhibits some monotonicity. The half circles B^+ and B^- are separated by line M. The optimal vector to choose against a vector in B^- like v_2 is a vector in B^+ like v_1. Notice that if v^* is not perpendicular to M, then v_1 is not diverging toward v^* but only away from M and v_2 as v_2 moves further from M into B^-. Hence v_2 does have some ability to draw v_1 away from v^*, but not

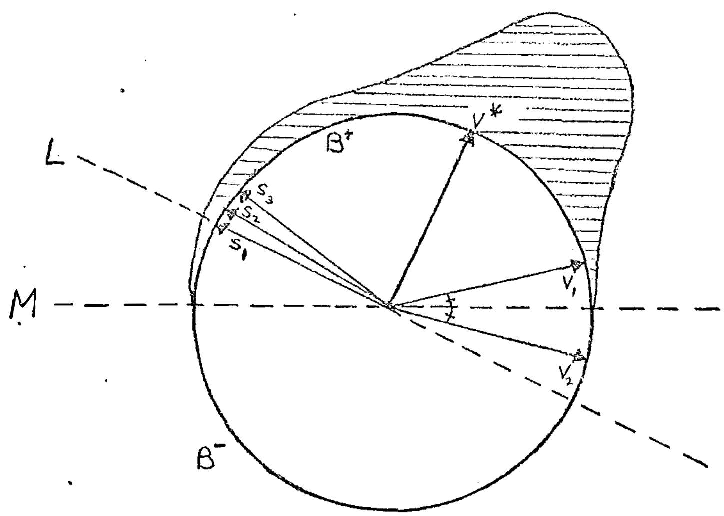

FIGURE 2
Exploiting a Fixed Opponent When Ideal Directions Are Distributed Monotonically

out of B^+. Furthermore, in this situation, if one candidate adopts a vector s_1 in B^+ that is not v^*, then his opponent increasingly receives more votes by choosing vectors increasingly closer to s_1, but always between s_1 and v^*. For example, s_2 does better than s_3 for candidate 1 when candidate 2 adopts s_1. Candidate 1 can insure that the fraction of the electorate voting for him is within any arbitrary amount of the fraction of the electorate whose preferred vectors lie above the line L.

IV. DIRECTION VOTING IN AN EUCLIDEAN MODEL

We now assume that individuals have well-defined preferences over an Euclidean issue space, but that the outcomes associated with the candidates are restricted to a small neighborhood of the status quo (origin). In fact, we assume the simplest case considered in Euclidean models: each voter i most prefers a point x_i and prefers y to z if and only if $\|x_i - y\| < \|x_i - z\|$. In the notation of the basic model, each voter i now most prefers the direction s_i for which $\lambda s_i = x_i$ has a solution $\lambda > 0$. When candidate j chooses a point strategy z_j, he is adopting the direction v_j for which $\lambda v_j = z_j$ has a solution $\lambda > 0$. Voting is again assumed to agree with issue preferences, and only indifferent individuals abstain.

The first question concerns the properties that a distribution of voter's preferred points must satisfy for plurality equilibria to exist. We first observe that if $\hat{P}$ is a probability measure representing preferred points in the issue space, it induces a probability measure P on B to represent preferred directions: $P[s \in A] = \hat{P}[x \in C(A)]$ for any (Borel) subset A of B, where $C(A) = \{x \in E^n: \alpha x \in A$ for some $\alpha > 0\}$ is the cone spanned by A, and $P[s = 0] = \hat{P}[x = 0]$. Thus the condition of theorem 2 can be considered to apply to $\hat{P}$ as well as P. But we show further that if an equilibrium point exists for a distribution of voters when candidates may choose any points in the issue space, then a corresponding undominated direction exists when outcomes associated with candidates are essentially shift directions. We first need formal definitions.

Definition 4: A point $z \in E^n$ is undominated provided $\hat{P}[\|z - x\| \leq \|y - x\|] \geq 1/2$ for all $y \in E^n$. A point z is an equilibrium in the plurality game provided it satisfies $\hat{P}[\|z - x\| < \|y - x\|] \geq \hat{P}[\|z - x\| > \|y - x\|]$ for all $y \in E^n$.

Definition 5: A point $z \in E^n$ is a total median of $\hat{P}$ provided $\hat{P}\,[a'(x - z) \geq 0] \geq 1/2$ for all $a \in E^n$.

As previously mentioned, an undominated point is known to be a total median. It is also true that, analogously to theorem 1, undominated points are equilibria.

Lemma 1: In an Euclidean model, $z \in E^n$ is undominated if and only if it is an equilibrium.

Proof: Let $y \in E^n$ and, for $0 \leq \alpha \leq 1$, define $y(\alpha) = \alpha y + (1 - \alpha)z$.

For z undominated, $\hat{P}[\|z - x\| \leq \|\alpha(y) - x\|] \geq \hat{P}[\|z - x\| > \|\alpha(y) - x\|]$.

$$\text{Hence, } \hat{P}[\|z - x\| < \|y - x\|] = \lim_{\alpha \to 1-} \hat{P}[\|z - x\| \leq \|\alpha(y) - x\|]$$

$$\geq \lim_{\alpha \to 1-} \hat{P}[\|z - x\| > \|\alpha(y) - x\|]$$

$$\geq \hat{P}[\|z - x\| > \|y - x\|].$$

Therefore z is an equilibrium. The converse is obvious.

Theorem 4: If $z \in E^n$ is an equilibrium point, then any direction v^* satisfying $\lambda v^* = z$ for some $\lambda \geq 0$ is undominated. If $z \neq 0$ or $\hat{P}[x = 0] = 0$, then v^* is also an equilibrium direction.

Proof: Let $v \in B$ be any direction except v^*. Then $z'(v^* - v) \geq 0$. Hence

$$P[s'(v^* - v) \geq 0] = \hat{P}[x'(v^* - v) \geq 0]$$

$$\geq \hat{P}[x'(v^* - v) \geq z'(v^* - v)] \geq 1/2$$

since z is undominated and hence a total median. This proves that v^* is undominated. If $z \neq 0$, then $z'(v^* - v) = \lambda(1 - v'v^*) > 0$ and

$$P[s'(v^* - v) > 0] = \hat{P}[x'(v^* - v) > 0]$$

$$\geq \hat{P}[x'(v^* - v) \geq z'(v^* - v)] \geq 1/2.$$

This implies that v^* is an equilibrium if $z \neq 0$. Finally, if $\hat{P}[x = 0] = 0$, then theorem 1 implies that v^* is an equilibrium.

The existence of undominated direction requires P to satisfy more than the median-like part of the condition in theorem 2, but theorem 4 establishes that no more than a total median condition on $\hat{P}$ is needed. In fact, the converse of theorem 4 is false -- existence of direction equilibria does not guarantee the existence of point equilibria for a corresponding Euclidean model. As a particularly easy example, illustrated in figure 3, suppose there are three voters whose ideal points P_1, P_2 and P_3 are arranged in a triangle to one side of the status quo S. Then no total median exists, but the direction vector v^* that points toward P_3 satisfies the condition of theorem 2 and so represents an equilibrium.[7]

A consequence of theorem 4 and corollary 1 is that the status quo is an equilibrium point if and only if all directions are undominated. Again, the heuristic interpretation is that the status quo is at a social maximum if and only if society is indifferent about the direction the status quo moves.

Theorem 4 also determines a consistency relationship between the two types of equilibria: if point equilibria exist, equilibrium direction vectors will "point" towards them. Suppose we now consider the situation in which a candidate may choose either a point or a direction as a strategy. Using another assumption about voter behavior, we can establish another consistency property for each type of strategy: if one candidate has chosen either a direction vector or a point (not the status quo) as his strategy, then his opponent can

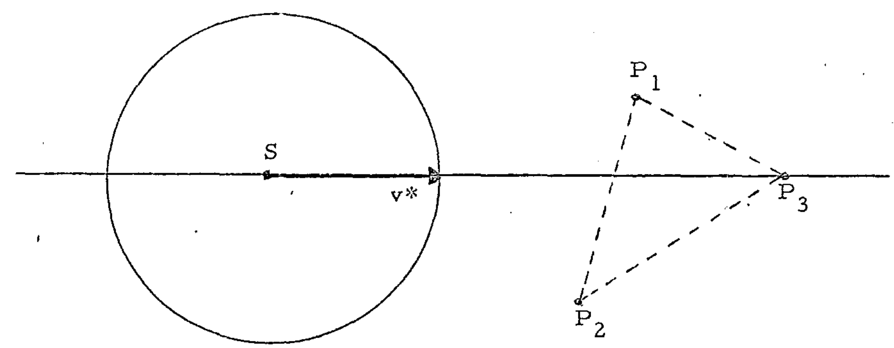

FIGURE 3

Situation with a Direction Equilibrium
But No Point Equilibrium

do no better than to choose the same type of strategy. The additional assumption concerns the voter's decision rule when one candidate chooses a direction and the other a point. We shall suppose the voter believes the candidate who chooses a point can shift the status quo the maintained distance, and that the other candidate would shift the status quo the same amount. Based upon an "equally likely" type of rationale, this assumption implies that voters will always vote as if the two candidates had chosen points on the same hypersphere about the status quo, i. e., voters will direction vote.

The internal consistency property now follows easily. If candidate 2 has chosen a direction, then regardless of the type of strategy candidate 1 chooses, the electorate will behave as if both had chosen directions. But if candidate 2 has chosen a point z, then for any direction v that candidate 1 might choose, he can achieve the same outcome by choosing the point $\|z\|v$.

However, the addition of either infinite or finite information costs might cause direction strategies to dominate point strategies. If the cost of obtaining information about voters' preferences over more than a neighborhood of the status quo is "too" high, or if only information about preferences over directions can be obtained, then the candidates will have no real basis for choosing point strategies. Possessing only uncertain knowledge about voter preferences away from the status quo, the risk-averse candidate may prefer a direction strategy to an exact point. If each candidate is

also uncertain as to the amount of information the other candidate possesses about voter preferences away from the status quo, it is even more likely that direction strategies will dominate. This follows because one candidate's choice of a direction strategy essentially forces the opponent to also choose a direction strategy and hence to utilize only information about the distribution of preferred directions, presumably known to both candidates. Formalization of these concepts is left for future work.

V. SUMMARY

The direction model of the electoral process allows limits to candidate mobility or voter perception and cognition. It is applicable (1) if only issue outcomes near the status quo are associated with candidates; (2) if only directional information is transmitted to voters; (3) if voter preferences are only well-defined near the status quo or are only defined for directions in which it can shift; or (4) if the outcome space is curved so that it can be modeled as a hypersphere.

Assuming that a voter will vote for the candidate who campaigns for a direction closest to his own preferred direction, plurality equilibria were shown to be undominated. The identity of the two types of solutions was established if nobody was totally indifferent. Then a necessary and sufficient condition for the existence of undominated directions was determined. The first part of the condition, stating that any hyperplane containing the undominated direction vector and the origin bisects the distribution of preferred directions, is analogous to the total median condition in the simple Euclidean models. The remainder of the condition in theorem 2, stating that a majority of the electorate's preferred direction vectors lie on the same side as the undominated direction vector of any hyperplane containing the origin, is not implied by the median-like property in this model because of the "curved" nature of the directional domain space. The second part of the condition is what allows a candidate to diverge from a fixed direction chosen by an extremist opponent, where at least half the feasible directions are defined to be extremist for every distribution of the electorates' preferred directions.

Although the addition of a second part to the characterizing condition for equilibrium seems to further decrease the likelihood of its occurrence, it was shown that in situations where the assumptions of the simple Euclidean model are met, point equilibria exist only if corresponding undominated directions also exist. But the converse of this theorem is false -- some distributions of voter preferences yield direction but not point equilibria. In situations where both types of equilibria exist, contradictory predictions will not occur since equilibrium direction vectors point in the direction of existing equilibrium points.

Finally, it was argued that a candidate has no incentive to adopt a type of strategy different from the type he knows his opponent will choose. This result can be interpreted as an internal stability property for each model. However, it was suggested that

when a candidate's uncertainty about voter's preferences away from the status quo and about the extent of his opponent's information are considered, only the direction model may exhibit this internal stability.

APPENDIX

Proof of Theorem 1: Suppose v^* is an equilibrium. Then for any other $v \in B$, $P[s'(v^* - v) > 0] \geq P[s'(v^* - v) < 0]$. Hence v^* is undominated since $P[s'(v^* - v) \geq 0] \geq P[s'(v^* - v) < 0] = 1 - P[s'(v^* - v) \geq 0]$.

Conversely, suppose v^* is undominated but not an equilibrium, and that $P[s=0] = 0$. For any $a \in E^n$ define the following sets:

$$S_1(a) = \{s \in B: s'a > 0\}$$
$$S_2(a) = \{s \in B: s'a < 0\}$$
$$H(a) = \{s \in B: s'a = 0\}.$$

By assumption, there exists $v \in B$ such that $PL_1(v^*, v) < 0$. Hence, letting $t = v^* - v$, there is an $\epsilon > 0$ such that

$$\text{(i)} \quad P[S_1(t)] < P[S_2(t)] - \epsilon$$

Since $P[\{a\}] > 0$ for only a countable number of $a \in B$, there exists $b \in B$ such that $b'v^* \geq 0$ and $P[H(b)] = 0$. Hence $P[H(b) \cap H(t)] = 0$. Let $H_i = S_i(b) \cap H(t)$ for $i = 1, 2$.

Consider the case $P[H_1] \leq P[H_2]$. For $n > 1$, define $c_n = n^{-1}b + (1 - n^{-1})t$. We now show that $\lim_{n\to\infty} S_i(c_n) = H_i \cup S_i(t)$, or, by definition, that

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_i(c_n) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_i(c_n) = H_i \cup S_i(t).$$

First, observe that $s'c_n = n^{-1}s'b + (1 - n^{-1})s't$ monotonically approaches $s't$ as $n \to \infty$. Hence $s \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_1(c_n) \iff s'c_n > 0$ for infinitely many $n \iff s'c_n \leq 0$ for only finitely many $n \iff s \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_1(c_n)$. Also, $s \in H_1 \cup S_1(t) \iff s't > 0$ or ($s't = 0$ and $s'b > 0$) $\iff s'c_n > 0$ for all n sufficiently large $\iff s \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_1(c_n)$. The argument for $i = 2$ is similar.

So by the continuity of a finite measure, and since H_i and $S_i(t)$ are disjoint, there exists an integer n_0 such that

$$\text{(ii)} \quad |P[S_i(c)] - P[H_i] - P[S_i(t)]| < \frac{\epsilon}{2}$$

for all c in the arc $A = \{c \in B: c = \alpha c_{n_0} + \beta t, \alpha > 0, \beta > 0\}$. For distinct $\hat{\hat{c}}, \hat{c} \in A$, there is no real number γ such that $\hat{\hat{c}} = \gamma\hat{c}$. Hence $s \in H(\hat{\hat{c}}) \cap H(\hat{c}) \iff s'b = 0$ and $s't = 0 \iff s \in H(b) \cap H(t)$. Thus $H(\hat{\hat{c}}) \cap H(\hat{c}) = H(b) \cap H(t)$ for all distinct $\hat{\hat{c}}, \hat{c} \in A$. So again by a countability argument, there exists $\bar{c} \in A$ such that $P[H(\bar{c})] = P[H(b) \cap H(t)] = 0$. From (i) and (ii), and since we are considering the case $P[H_1] \leq P[H_2]$, we now obtain

$$\begin{aligned}\text{(iii)} \quad P[S_1(\bar{c})] &< P[H_1] + P[S_1(t)] + \frac{\epsilon}{2} \\ &< P[H_2] + P[S_2(t)] - \epsilon + \frac{\epsilon}{2} \\ &< P[S_2(\bar{c})].\end{aligned}$$

Now if $v^* = 0$, let $\bar{v} = -\bar{c}$, and otherwise let $\bar{v} = v^* - (2\bar{c}'v^*)\bar{c}$. Clearly, $\bar{v} \in B$. Furthermore, since $b'v^* \geq 0$ and $v'v^* < 1$, when $v^* \neq 0$ we have

$$\bar{c}'v^* = (\alpha n_0^{-1})b'v^* + [\alpha(1 - n_0^{-1}) + \beta](1 - v'v^*) > 0.$$

Thus $P[s'(v^* - \bar{v}) \geq 0] = P[s'\bar{c} \geq 0]$, whether or not $v^* = 0$. Hence, as $P[H(\bar{c})] = 0$, (iii) implies that

$$P[s'(v^* - \overline{v}) \geq 0] = P[S_1(\overline{c})] + P[H(\overline{c})]$$
$$< P[S_2(\overline{c})]$$
$$= 1 - P[s'(v^* - \overline{v}) \geq 0].$$

Therefore v^* is not undominated, contrary to assumption. The proof is similar for the case $P[H_1] > P[H_2]$.

FOOTNOTES

1. The assumptions of electcral spatial models and many of their predicticns are reviewed in Davis, Hinich, and Ordeshook (1970), and Riker and Ordeshook (1973).

2. In this interpretation the status quo must be on the surface of the hypersphere rather than at its center. The status quo shall be assumed under this interpretation to play no role in the model, just as it plays no role in the usual Euclidean spatial models.

3. Empirical evidence for direction voting might be found in some spatial model experiments conducted by Fiorina and Plott (personal communication -- but see Fiorina and Plott [1975] for details on similar experiments). In their experiments, each voter's payoff function declined with distance from a single point where it achieved its maximum. When a candidate asked: "Who wants me to move into this rectangle?", usually all voters whose optimal points were in the specified rectangle indicated approval of the move. If the voters had utilized subjective estimates of the distances the candidate would move into the specified rectangle, those voters very near the border containing

the candidate's current position probably would not have been in favor of such a move. But as it turned out, most who had a utility gradient at the candidate's current point that formed an acute angle with the proposed direction vector favored the move. This behavior suggests direction voting.

4. See Matthews [1977] for an extensive treatment of directional preferences.

5. Undominated directions to simple games with a finite number of players are discussed extensively in Matthews [1977].

6. Total medians are formally defined in definition 5, section IV.

7. However, it is shown in Matthews [1977] that existence of undominated directions is equivalent to satisfaction of pairwise symmetry conditions similar to those Plott (1967) establishes for his constrained voting equilibria. Their stringency implies that existence is only slightly more "common" for directional than point equilibria. Cohen and Matthews [1977] elaborate on this point.

8. When an individual prefers z_1 over z_2 if and only if $\| x_i - z_i \| < \| x_i - z_2 \|$, direction voting exactly agrees with preferences if the outcomes candidates can choose are constrained

to lie on the same hypersphere centered at the status quo (origin). This follows trivially for $x_i = 0$. Otherwise, if $\| z_1 \| = \| z_2 \|$, $\| x_i - z_1 \| < \| x_i - z_2 \| \iff x_i' z_1 > x_i' z_2 \iff s_i' v_1 > s_i' v_2$, where $s_i = x_i / \| x_i \|$ and $v_j = z_j / \| z_j \|$.

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